  **REAL ANALYSIS**

**UNIT I**

**Sec 4 : ORDER IN R:**

 **O1: LAW OF TRICHOTOMY:**

**Given any two real numbers a, b, one and only one of the following holds:**

**a** $> $**b, a = b, b**$> $**a.**

**O2: TRANSITIVITY:**

**For each triple real numbers a, b, c, if**

**a** $>$**b, b** $> $**c , then a** $>$ **c**

**O3: MONOTONE PROPERTY FOR ADDITION :**

**For all real numbers a, b and c, a**$>$**b it implies a+c** $> $**b+c**

**O4 : MONOTONE PROPERTY FOR MULTIPLICATION :**

**For all real numbers a,b and c, a**$>$**b and**

**c**$>$ **0 it implies ac** $>$ **bc**

**SECTION 4.1 : POSITIVE NUMBERS**

**Definition 4.1**

**A real number a is said to be positive if a** $>$ **0.**

**THEOREM 4.1. For each real number a one and only one of the following holds:**

 **a** $>$ **0, a =0, -a** $>$ **0**

**PROOF**

**In view of O1 , it suffice to prove that**

 **0** $>$ **a ⇔ -a** $>$ **0.**

**Now**

 **0** $>$ **a ⇒ 0 +(-a)** $>$ **a+(-a) ⇔ -a** $>$ **0**

 **-a** $>0 $**⇒ (-a) +a** $>$ **0 + a ⇒ 0** $>$ **a**

**THEOREM 4.2. If a, b , be positive real numbers , then a+b is a positive real number.**

**PROOF**

  **a** $>$ **0 ⇒ a +b** $>$ **0+b, (By O3)**

 **⇒ a+b** $>$ **b** $>$ **0 (Since b**$>$ **0)**

**a+b >0**

**THEOREM 4.3. If a,b be positive real numbers , then ab is a positive real number.**

**PROOF**

  **a** $>$ **0 and b** $>$ **0 ⇒ ab** $>$ **0.b (By O4)**

 **But 0.b=0.Therefore , we have ab** $>$ **0.**

**SECTION 4.2.**

**THE ORDER RELATIONS** $<,\geq .\leq $

**DEFINITION 4.2**

**A real number a is said to be less than b (written a** $<$ **b) if b** $>$ **a**

**A real number a is said to be negative if a** $<$ **0**

**THEOREM 4.4 .Given any two real numbers a,b, one and exactly one of the following holds:**

  **a** $< $**b, a=b, b** $<$ **a.**

**THEOREM 4.5. For all real numbers a,b and c,**

 **a**$<$ **b ⇔ a+c** $<$ **b+c**

 **a** $<$ **b and c** $<$ **0 ⇔ ac** $>$ **bc.**

**THEOREM 4.6. For each real number a, one and only one of the following holds:**

**a** $<$ **0, a=0, -a** $<$ **0.**

**THEOREM 4.7.**

**a** $<$ **0, b** $<$ **0 ⇔ a +b** $<$ **0 and ab** $>$ **0.**

**DEFINITION 4.3:**

**A real number a is said to be greater than or equal to b (written a** $\geq $ **b) if either**

 **a** $>$ **b or a=b.**

**A real number a is said to be less than or equal to b (written a** $\leq b$ **) if either**

 **a** $<$ **b or a=b.**

**The relations** $ \geq $ **and** $\leq $ **are called weak inequalities and the relations** $<$ **and** $>$ **which are called the strict inequalities.**

**SECTION 5. ABSOLUTE VALUE**

**DEFINITION 5.1**

**If x be a real number , then its absolute value denoted by** $\left|x\right|$ **, is defined by the rule**

$\left|x\right|=\left\{\begin{array}{c}-x, if \&x<0\\x, if \&x\geq 0\end{array}\right.$

**We may observe that** $\left|x\right|$ **is defined for every x ϵ R.**

**Also x1= x2 ⇒** $\left|x\_{1}\right|$ **=**$\left|x\_{2}\right|$ **.**

**THEOREM 5.1 .For every x ϵ R.**

$\left|x\right|$ **= max {-x, x}.**

**PROOF**

**By the law of trichotomy ,one and exactly one of the following is true**

 **(i) x** $>$ **0, (ii) x =0, (iii). x**$ <$ **0**

**If** $x\geq 0$ **,then** $\left|x\right|$ **=x, and x** $\geq $ **-x**

**If x** $<0$ **,then** $\left|x\right|$ **= -x , and -x** $>$ **x**

**Thus in either case ,** $\left|x\right|$ **is the greater of two numbers x and –x , that is**

$\left|x\right|$ **= max {-x, x}.**

**COROLLARY: For every x ϵ R, x** $\leq $$\left|x\right|.$

**PROOF**

 $\left|x\right|$ **= max {-x, x}** $\geq $ **x.**

**THEOREM 5.2 For every x ϵ R ,**

$ \left|x\right| $**2 = x2 =** $\left|-x\right| $**2**

**PROOF**

**By definition ,**

$\left|x\right|=\left\{\begin{array}{c}-x, if \&x<0\\x, if \&x\geq 0\end{array}\right.$

**In either case ,**

$\left|x\right| $**2 = x2 .**

**Also similarly**

$\left|-x\right| $**2 = (-x)2 = x2 .**

**Hence** $ \left|x\right| $**2 = x2 =** $\left|-x\right| $**2 .**

**THEOREM 5.3. For every x ϵ R,** $\left|x\right|=$$\left|-x\right|$**.**

**PROOF**

$\left|-x\right|$ **= max {-x, -(-x)},**

 **= max {-x, x},**

 **=** $\left|x\right|$

**THEOREM 5.4, For all x,y ϵ R ,**

$\left|x.y\right|$ **=** $\left|x\right|$ **.**$ \left|y\right|$ **.**

**PROOF**

 $\left|x.y\right| $**2= (xy)2 ,**

 **= x2 y2,**

**=**$ \left|x\right| $**2** $ \left|y\right| $**2**

 **=(**$\left|x\right| $$. \left|y\right| $**)2 .**

**Since** $\left|x.y\right|$ **=** $\left|x\right|$ **.**$ \left|y\right|$ **are both non negative , therefore taking the positive square roots of both sides ,we have** $\left|x.y\right|$ **=** $\left|x\right|$ **.**$ \left|y\right|$**.**

**THEOREM 5.5 (THE TRIANGLE INEQUALITY).**

**Statement:**

**For all real numbers x and y,**

$\left|x+y\right|$ **≤** $\left|x\right|$ **+**$ \left|y\right|$

**PROOF**

**CASE 1. x+ y ≥ 0.**  **In this case**

$\left|x+y\right|$ **= x+y**

 **Since x ≤** $\left|x\right|$ **and y≤** $\left|y\right|$**, therefore it follows that** $x+y$ **≤** $\left|x\right|$ **+**$ \left|y\right|$**,**

**and consequently** $\left|x+y\right|$ **≤** $\left|x\right|$ **+**$ \left|y\right|$

**CASE 2. x+y** $<$ **0. In this case**

 **-(x+y)**$>$ **0, that is (-x)+(-y)** $>$ **0.**

 **Now** $\left|x+y\right| $**=** $\left|-(x+y)\right|$**,**

 **=** $\left|\left(-x\right)+(-y)\right|$

 **≤** $\left|-x\right|$ **+**$ \left|-y\right|$

 **Since** $\left|-x\right|$ **=**$\left|x\right|$ **,**$ \left|-y\right|$ **=** $\left|y\right|$**, therefore it follows that** $\left|x+y\right|$ **≤** $\left|x\right|$ **+**$ \left|y\right|$**.**

**THEOREM 5.6 .**

**For all real numbers x and y ,**

$ \left|x-y\right|\geq $$\left|\left|x\right|-\left|y\right|\right|$**.**

**PROOF**

**By the triangle inequality ,we have**

$\left|x\right|$ **=** $\left|(x-y)+y\right|$

 **≤** $\left|x-y\right|$ **+**$ \left|y\right|$

**So that** $\left|x\right|$ **-** $\left|y\right|$ **≤** $\left|x-y\right|$ **---------(1)**

$ Again \left|y\right|$ **=** $\left|(y-x)+x\right|$

 **≤** $\left|y-x\right|$ **+**$ \left|x\right|$

**So that** $\left|y\right|$ **-** $\left|x\right|$ **≤** $\left|y-x\right|$

**i.e -(** $\left|x\right|$ **-** $\left|y\right|$ **) ≤** $\left|x-y\right|$**,**

**since** $\left|y-x\right|= \left|x-y\right|$ **--------(2)**

 **Now** $\left|\left|x\right|-\left|y\right|\right|$ **= max{**$\left|x-y\right|$**, -(** $\left|x-y\right|$**)}**

 **≤** $\left|x-y\right|$ **, by (1) and (2)**

**EXAMPLE 1**

**If x , y be any real numbers , show that**

$\left|x+y\right| $**2 +** $\left|x-y\right| $**2 = 2(**$\left|x\right| $**2 +** $ \left|y\right| $**2 ).**

**SOLUTION**

$\left|x+y\right| $**2 +** $\left|x-y\right| $**2 = (x+y)2 + (x-y)2**

 **= 2 (x2 +y 2),**

 **= 2(**$\left|x\right| $**2 +**$\left|y\right| $**2 )**

**EXAMPLE 2:**

**If x, l,** $\in $ **be real numbers, and** $\in $$>$ **0, show that**

$\left|x-l\right|<\in $ **⇔ l-**$\in <x$ **<l+** $\in $

**SOLUTION**

$\left|x-l\right|<\in $ **⇔ max {(x-l), -(x-l)} <** $\in $**,**

 **⇔ x-l <** $\in $ **and l-x <** $\in $

 **⇔ x<l+**$\in $ **and l-**$\in $ **<x**

 **⇔ l-**$\in <x$ **<l+** $\in $

**SECTION 6**

**COMPLETENESS**

**DEFINITION 6.1.**

**If for a set S of real numbers , there exists a real number u, such that**

 **X** $\in $**S ⇒ x ≤ u,**

**then u is called an upper bound of S .If there exists an upper bound for a set S , then S is said to be bounded above.**

**ILLUSTRATIONS:**

**1. The set of negative real numbers is bounded above ,0 being an upper bound.**

**2. The set of positive real numbers is not bounded above . For , if we assume that the set R+ of positive real numbers is bounded above ,and that u is an upper bound ,then we are immediately led to a contradiction by observing that**

**(i) since 1**$\in $ **R+ , therefore ,1≤ u, which means u>0.**

**(ii) u+1 >0 and consequently**

 **u+1** $\in $ **R+ ;**

**(iii) u < u+1, so that there is an x in R+ , namely u+1 , which does not satisfy x ≤ u**

**(iv) since u is assumed to be an uppr bound R+ ,therefore ,we should have x ≤ u for all x in R+.**

**From illustrations 2 above , we find that it is not necessary that a set should be bounded above. However , if a set has one upper bound , then it has many upper bounds . For , if u be an upper bound of a set S, then every real number u’ greater than u is also an upper bound of S.**

**DEFINITION 6.2.**

**If the set of all upper bounds of a set S of real numbers has a smallest member , say w, then w is said to be a least upper bound or a supremum of S (written sup S).**

**It can be easily seen that a set cannot have more than one supremum. In fact , if w, w’ be two suprema of a set S , then**

**( i) w and w’ are both upper bounds of S ;**

**( ii) since w is a supremum of S and w’ is an upper bound of S , therefore , w ≤ w’. that is**

 **, w w’;**

**(iii) since w’ is a supremum of S and w is an upper bound of S ,therefore ,**

 **w’≤ w, that is w’ w:**

**(iv) by the law of trichotomy w w’ and**

**w’ w together imply w=w’.**

**ORDER COMPLETENESSS PROPERTY**

 **Every non empty set of real numbers which is bounded above has a supremum.**

**THEOREM 6.1**

**ARCHIMEDEAN PROPERTY OF REAL NUMBERS**

**Statement**

**If x and y be any positive real numbers , then**

 **there exists a positive integer n such that**

**ny >x.**

**PROOF**

**( i) Suppose the statement of the theorem is false.**

**( ii) Then , for each positive integer n , we must ny ≤ x.**

**(iii) This means that x is an upper bound of the set**

 **S = {y, 2y, 3y,….}**

**(iv) By the completeness property of R , S must have a supremum, say ‘s’.**

**(v) Then ,ny ≤ s for all positive integers n, and consequently (n+1)y ≤ s for all positive integers n.**

**(vi) This implies that ny ≤ s-y for all positive integers n, so that s-y is an upper bound of S .**

**(vii) Thus , we have an upper bound of S , namely s-y , which is less than the supremum of S .**

**(viii) Since this contradicts the definition of s , therefore , the statement of the theorem must be true.**

**COROLLARIES.**

 **1.If x be any real number , then there exists a positive integer n such that n > x.**

**PROOF**

**Take y=1 in the proof of the theorem.**

**2. If x be any real number , and y be any positive real number , then there exists a positive integer n such that ny > x.**

**PROOF**

**If x >0 , the corollary is a re statement of theorem 6.1. If x ≤ 0, then n=1 suffices .**

**For , 1.y =y >0 ≥ x.**

**3. If x be any real number , then there exists a positive integer n such that n>x.**

**PROOF**

**Take y=1 in corollary 2.**

**THEOREM 6.2.**

**CHARACTERISATION OF THE SUPREMUM OF A SET**

**STATEMENT:**

**Let S be a non empty set of real numbers bounded above. Then a real number s is the supremum of S iff the following two conditions hold:**

**( i) x ≤ s for all x**$\in $ **S.**

**( ii) For each positive real number** $\in $ **, there is a real number x** $\in $ **S that x> s-**$\in $**.**

**PROOF**

**\*The conditions are necessary.**

**\*In fact , since s is the supremum of S ,therefore , for all x**$\in $ **S, we must have x ≤ s.**

**\*Also , if x be any positive number whatever , then s-**$\in $ **cannot be an upper bound of S**

 **(for it is less than the supremum) and therefore , for some x**$\in $ **S , we must have**

**x > s-** $\in $**.**

**\*The conditions are sufficient as well.**

**\*Suppose there exists a real number s’ satisfying the conditions (i) and (ii).**

**\*By (i) it follows that s is an upper bound of S \*Also if s’ is any real number less than s , then s-s’ >0.**

**\*Letting** $\in $ **= s- s’, we find by (ii) that there exist an x** $\in $ **S such that x> s-** $\in $ **,**

**(i.e) x >s’, showing that s’ is not an upper bound of S .**

**\*Thus we find that s is an upper bound of S and any number less than s is not an upper bound of S .**

**\*H ence s is the supremum of S.**

**SECTION 6.1**

**LOWER BOUNDS**

**DEFINITION 6.3**

**If for a set S of real numbers , there exists a real number v such that**

 **x**$\in $ **S ⇒ x ≥ v,**

**then v is called a lower bound of S . If there exists a lower bound for the set S , then S is said to be bounded below.**

**ILLUSTRATIONS**

**1.The set of positive real numbers is bounded below , 0 being a lower bound .**

**2.The set of negative real numbers is not bounded below**

**From illustration 2 above , we find that it is not necessary that a set should be bounded below .However , if a set has one lower bound , then it has many lower bounds . For , if v be a lower bound of a set S , then every real number v’ less than v is also a lower bound of S.**

**DEFINITION 6.4**

**If the set of all lower bounds of a set S of real numbers has a greatest member , say t, then t is said to be a greatest lower bound or an infimum of S ( written inf S).**

**THEOREM 6.3**

**Any non empty set of real numbers which is bounded below has an infimum.**

**PROOF**

**\*Let S be any non empty set of real numbers and let v be a lower bound of S.**

**\*Let us denote by T the set of non negatives of members of S.**

**\*That is , T = { -x : x**$\in $ **S }.**

**\*We shall show that T is bounded above .**

**\*In fact, if y be an arbitrary member of T , then y = -x for some x**$\in $ **S.**

**\*Since v is a lower bound of S , therefore , it follows that x ≥ v, and consequently y≤ -v.**

**\*Since y≤ -v for all y**$\in $ **T ,therefore T is bounded above , -v being an upper bound .**

**\*By the completeness property , T has a supremum , say t.**

**\*It can be shown that -t is the infimum of S.**

**\*This is equivalent to showing that if w be any lower bound of S , then -t ≥ w.**

**\*Now , w is a lower bound of S ⇒ -w is an upper bound of T ⇒ t ≤ -w ⇒ -t ≥ w.**

**\*Hence the theorem.**

**THEOREM 6.4**

**CHARACTERISATION OF THE INFIMUM OF A SET**

**STATEMENT**

**Let S be a non empty set of real numbers bounded below. Then a real number t is the infimum of S iff the following two conditions hold:**

**( i) x ≥ t for all x**$\in $ **S.**

**( ii) For each positive real number** $\in $ **, there is a real number x** $\in $ **S that x< t +**$\in $**.**

**PROOF**

**\*The conditions are necessary.**

**\*In fact , since t is the infimum of S ,therefore , for all x**$\in $ **S, we must have x ≥ t.**

**\*Also , if** $\in $ **> 0 be given , then t +**$ \in $ **is greater than the infimum of S and cannot therefore , be a lower bound of S.**

 **\*This implies that , for some x**$\in $ **S , we must have x < t+**$\in $**.**

**\*The conditions are sufficient as well.**

**\*Suppose there exists a real number t’ satisfying the conditions (i) and (ii).**

**\*By (i) it follows that t is a lower bound of S \*Also if t’ is any real number greater than t , then t – t’> 0.**

**\*Letting** $\in $ **= t’ -t, we find by (ii) that there exist an x** $\in $ **S such that x< t +**$\in $ **,**

**(i.e) x < t’, showing that t’ is not an lower bound of S .**

**\*Thus we find that t is a lower bound of S and no number greater than t is not a lower bound of S .**

**\*H ence t is the infimum of S.**

**SECTION 10**

**COUNTABLE AND UNCOUNTABLE SETS**

**DEFINITION 10.1**

**A set S is said to be finite if either it is empty , or for some natural number n, there exists a one to one mapping from the set {1,2,….n} onto the set S. If a set is not finite , then it is said to be infinite.**

**ILLUSTRATIONS**

**1.The set** $∅$ **is a finite set.**

**2.The set {e,**$π$ **,** $\sqrt{2}$ **} is a finite set, because their exist several one to one mappings from the set {1,2,3} onto the set {e,**$π$ **,** $\sqrt{2}$ **} ,one such mapping being 1→ e, 2→** $π$**, 3→**$\sqrt{2}$

**3.The set of all primes less than 10100 is a finite set.**

**4.The set of all human beings in the world at a particular instant is a finite set.**

**5.The set of all natural numbers is an infinite set.**

**6.The set of all rational numbers is an infinite set.**

**7.The set { x: x**$\in $ **R and and 0 ≤ x ≤ 1} is an infinite set.**

**THEOREM 10.1**

**(a)Every subset of a finite set is a finite set.**

**(b)Every superset of an infinite set is an infinite set.**

**(c)The intersection of every non empty family of finite sets is a finite set.**

**(d)The union of every non empty family of infinite sets is an infinite set.**

**DEFINITION 10.2**

 **A set S is said to be enumerable, if there exists a one to one mapping from the set N of all natural numbers onto the set S.**

 **A set S is said to be countable if it is either finite or enumerable. If a set is not countable, then it is said to be uncountable.**

**ILLUSTRATIONS**

**1.The set N of all natural numbers is enumerable , the identify mapping being a desired one-to-one mapping.**

**2.The empty set is countable.**

**3.The set {4, -7, e,** $π\sqrt{5}$ **} is a countable set.**

**4.The set Z of al integers is a countable set. For , by rearranging the integers , we may write Z as { 0, -1, 1, -2, 2, -3, 3…..} .Let now f be a function from N to Z , defined by**

 **f(n) =** $\frac{1}{2}$ **(n-1), for n= 1,3, 5,….**

 **f(n) =** $-\frac{1}{2}$ **n , for n= 2,4,6,…**

**It can be easily seen that f is univalent as well as onto.**

**THEOREM 10.2**

**Every subset of a countable set is countable.**

**PROOF**

**\*Let A be a countable set and let B be a subset of A .**

**\*If B is finite , we have nothing to prove .**

**\*We may , therefore , assume without loss of generality that A is an infinite countable set and that B is an infinite subset of A.**

**\*Let A = { a1 ,a2 ,a3,… }.**

**\*Each element of B is an a1, for some index**

**1.**

**\*Let n1 be the smallest index for which**

$a\_{n\_{1}}$$\in $ **B.**

**\*Consider now the set A** $\~$ **{**$a\_{n\_{1}}\}$**.**

**\*Let n2 be the smallest index for which** $a\_{n\_{2}}$ **belongs to B as well as to A** $\~$ **{**$a\_{n\_{1}}\}$**.’**

**\*Consider now the set A** $\~$ **{**$ a\_{n\_{1}}, a\_{n\_{2}}\}$**.**

**\*Let n3 be the smallest index for which** $a\_{n\_{3}}$ **belongs to B as well as to A** $\~$ **{**$ a\_{n\_{1}}, a\_{n\_{2}}\}$**.**

**\*Proceeding in this manner , we find that**

**B = {** $a\_{n\_{1}}, a\_{n\_{2} }, a\_{n\_{3}},…\}$**.**

**\*Then k→**$a\_{n\_{k}}$ **is a one to one function from N onto B , and consequently B is countable.**

**THEOREM 10.3**

**Every superset of an set uncountable set is uncountable.**

**PROOF**

**\*This theorem is simply the dual of theorem 10.2.**

**\*Let A be an uncountable set and let B ⊃ A .**

**\* If B is countable , then the set A must also be countable (For , it is a subset of the countable set B ).**

**\*Since A is given to be uncountable , it follows B must also be uncountable.**

**THEOREM 10.4**

**If A1, A2,… are countable sets , then** $\bigcup\_{n=1}^{\infty }A\_{n}$ **is countable.**

**PROOF**

**\*Let us write**

 **A1= {a11, a12, a13, a14, …..},**

 **A2= {a21, a22, a23, a24, …..},**

 **A3= {a31, a32, a33, a34, …..},**

 **. . . . .**

 **. . . . .**

 **. . . . .**

 **An= {an1, an2, an3, an4, …..},**

 **. . . . .**

 **. . . . .**

 **. . . . .**

**\*Here aij stands for the jth element of the ith set as listed above.**

**\*Let us define the height of the element aij, to be i+j.**

**\*With this definition , the height of a11, is 2, and this is the only element of height 2.**

**\*Similarly , the height of each of the elements a12 and a21 is 3 and these are the only elements of height 3.**

**\*Similar observations can be made about other elements .**

**\*Since each element will have a unique height therefore , we can arrange the elements according to their heights as**

**a11, a12, a21, a31, a22, a13,…… leaving out any element that has already occurred.**

**\*Thus all elements will be counted out and consequently** $\bigcup\_{n=1}^{\infty }A\_{n}$ **is countable.**

**\*The following diagram gives a visual picture of the above counting process.**

**a11 a12 a13 a14  ……**

**a21 a22 a23 a24 ……**

**a31 a32 a33  a34 ……**

**a41 a42 a43 a44 ……**

**… … … … …..**

 **According to this scheme , the element are to be counted as**

**a11, a12, a21, a31, a22, a13, a14, a23, a32, a41,……**

**THEOREM 10.5**

**The set N**$×$ **N is countable.**

**PROOF**

**We may arrange the set N**$×$ **N as shown in fig**

**(1,1) (1,2)(1,3) (1,4) ……**

**(2,1) (2,2) (2,3) (2,4) ……**

**(3,1) (3,2)(3,3) (3,4) ……**

**(4,1) (4,2) (4,3) (4,4) ……**

**… … … … …..**

**This scheme arranges all the elements of**

**N**$×$ **N into a sequence and consequently shows that N**$×$ **N is countable.**

**CORALLARIES**

**1.The set of all positive rational numbers is countable.**

**PROOF**

**\*Every positive rational number is expressible as p/q, where p and q are positive integers prime to each other.**

**\*Let us denote the set of all positive rational numbers by A and let B be the set defined as**

 **B = {(p,q) : (p,q)**$\in $ **N**$×$ **N , p and q are prime to each other}**

**\*It is obvious that the elements of A and B are in one to one correspondence , and therefore , A is countable if and only if B is countable.**

**\*Since the set B is a subset of the countable set N**$×$ **N , therefore , it is countable.**

**\*Hence A is countable.**

**2.The set of all negative rational numbers is countable.**

**PROOF**

**\*The set C of all negative rational numbers can be put in one to one correspondence with the set A of all positive rational numbers.**

**\*Since A is countable , therefore , C must be countable.**

**3. The set Q of all rational numbers is countable.**

**PROOF**

**\*The set Q is the union of three countable sets A, C and {0}.**

**(since corollaries 1 and 2).**

 **The above corollary can also be proved as a direct consequence of theorem 10.4.**

**PROOF**

**\*For each natural number n , let**

 **An =** $\left\{\frac{0}{n},\frac{-1}{n},\frac{1}{n},\frac{-2}{n},\frac{2}{n},……\right\}$ **.**

**\*Then it can be easily seen that An  is countable.**

**\*Now** $\bigcup\_{n=1}^{\infty }A\_{n}$ **is a countable union of countable sets , and therefore , by theorem 10.4 it is countable.**

**\*But this union is precisely the set of all rational numbers.**

**\*Hence the set of all rational numbers is countable.**

**4. The set of all rational numbers in [0,1] is countable.**

**PROOF**

**\*The set of all rational numbers in [0,1]is a subset of the set of rational numbers which is countable.**

**\*Therefore , by theorem 10.2.,the set of all rational numbers in [0,1] is countable.**

**\*We shall now prove an important result which says that the set of all real numbers is uncountable.**

**\*In view of theorem 10.3, it is enough to show that the set [0,1] of all real is uncountable.**

**\*For this purpose we shall assume that every real number x can be expressed in decimal form as**

 **x= a0 . a1 a2 a3 ….,**

 **= a0 +** $\frac{a\_{1}}{10}+\frac{a\_{2}}{10^{2}}+\frac{a\_{3}}{10^{3}}+…,$

**where , a0 is an integer , and a1 , a2 , … are all integers such that 0≤ ai ≤ 9 , for all i.**

**\*This expression for a real number is unique except when the real number is a rational number of the form** $\frac{p}{2^{n}5^{n}}$ **, where p is an integer and m and n take any of the value**

**0, 1, 2, 3,…..**

**\*In such a case , two decimal expansions are possible .**

**\*For example, we may express** $\frac{1}{2}$ **either as 0.5000… or as 0.4999…**

**\*Conversely , every decimal of the form**

 **a0 . a1 a2 a3…. Are all integers , a1 ,a2 ,a3…. are all integers such that 0≤ ai ≤ 9 , for all i, is the decimal expansion of some real number.**

**THEOREM 10.6.**

**The set [0,1] is countable.**

**PROOF**

**\*We are now ready to show that the set [0,1] is uncountable.**

**\*Suppose that [0,1] is countable.**

**\*Then there exists a one to one mapping from N onto [0,1].**

**\*This means that if this mapping be f, then the set [0,1[ can be written as**

 **{ f(1), f(2),f(3),….f(n),…}.**

**\*Expressing each f(n) as a decimal , we have**

 **f(1) = 0. a11 a21 a31….,**

 **f(2) = 0. a12 a22 a32….,**

 **f(3) = 0. a13 a23 a33….,**

 **………………………,**

 **f(n) = 0. a1n a2n a3n….,**

 **……………………….**

**all the aij** $'$ **s being integers belonging to the set {0,1,2,3,…..9}.**

**\*Let us choose for each n** $\in $ **N, Aa positive integer bn as follows :**

 **bn = 1 if ann**$\ne $ **1,**

 **bn = 2 if ann =1.**

**\*That is , if a11=1,we choose b1=2 and if**

**a11** $\ne $ **1, we choose b1=1, and like wise for**

**b2 , b3 ,…..**

**\*This choice means that for each n, bn**$ \ne $ **ann**

**\*Let now y = 0. b1 b2 b3…..**

**\*Now y is a real number in [0,1].**

**\*Also , it is not in the set {f(1),f(2),….}.**

**\*Infact , it differs from f(1) in the first decimal place because b1**$ \ne $ **a11 , it differs from f(2) in the second decimal place ,….it differs from f(n) in the nth decimal place,….**

**\*Also , the decimal expansion of y is unique , since no bn is equal to 0 or 9.**

**\*This means that y**$ \ne $ **f(n) for any n .**

**\*We have thus found a real number y which is in [0,1], but which is not in {f(1),f(2),,…f(n),…}.**

**\*This contradicts the assumption that the set [0,1] is countable.**

**\*Hence the set [0,1] is countable.**

**COROLLARIES**

**1.The set of real numbers is uncountable.**

**2.The set of irrational numbers is uncountable.**

**PROOF**

**\*Let S be the set of irrational numbers.**

**\*If S be countable, then the set S**$∪$**Q (where Q is the set of rational numbers) will be countable.**

**\*S U Q= R and since the set R is uncountable , therefore , we have a contradiction.**

**\*Hence the set S is uncountable.**

**THEOREM 10.7**

**Let Pn be the set of polynomial functions f of degree n defined by relations of the form**

 **f(x) = a0xn + a0xn-1+….+an,**

**where n is a fixed non negative integer , the coefficients a0 a1……. an are all integers and a0**$\ne $**0.The set Pn is countable.**

**PROOF**

**\*We shall prove the result by induction on n, the degree of f.**

**\*The result is true for n=0.**

**\*For, the set of all polynomials of degree zero is one to one correspondence with the set Z**$\~$**{0} of a non zero integers and is, therefore , countable.**

**\*Let us now assume that the set Pk is countable for some fixed positive integer k.**

**\*For each positive integer m, let**

 **Sm= {f: f=mxk+1+g, g**$\in $ **Pk}**

 **S-m= {f: f= -mxk+1+g, g**$\in $ **Pk}**

**\*The sets Sm and S-m are both countable , each being in one to one correspondence with the countable set Pk.**

**\*Since the union of two countable sets is countable , therefore ,the set Tm= Sm**$∪$ **S-m is countable.**

**\*Again , since the union of two countable family of countable sets is countable, therefore ,** $\bigcup\_{m=1}^{\infty }T\_{m}$ **is countable.**

**\*Since Pk+1=**$\bigcup\_{m=1}^{\infty }T\_{m}$**, therefore Pk+1 is countable.**

**COROLLARY**

**The set P of polynomial functions with integer coefficients is countable.**

**PROOF**

**\*If Pn be the set of polynomial functions of a degree n with integral coefficients , then Pn is countable.**

**\*Since P =** $\bigcup\_{n=0}^{\infty }P\_{n}$ **,and since the union of countably many countable sets is countable;therefore , it follows that P is countable.**

**REMARK**

1. **For each fixed non negative integer n , the set Qn of polynomial functions of the form**

 **a0xn + a0xn-1+….+an,**

 **where a0 a1……. an are rational numbers**

 **and a0**$\ne $ **0, is countable.**

**(2)If Qn be as in (1) above , then**

$\bigcup\_{n=0}^{\infty }Q\_{n}$ **is countable.**

**DEFINITION10.3.**

**A real number is said to be algebraic if it is the root of some polynomial equation with rational coefficients.**

**THEOREM 10.8.**

**The set of algebraic numbers is countable.**

**PROOF**

**\*Let n be an arbitrary but fixed positive integer .**

**\*The set Qn is countable.**

**\*We may , therefore , write it as**

**{fn1, fn2,….}, where each fnk, is a polynomial of degree n with rational coefficients.**

**\*If Ank denotes the set of real roots of the equation fnk=0, then Ank is a countable set (in fact , it is a set consisting of atmost n elements).**

**\*Let** $\bigcup\_{k=1}^{\infty }A\_{nk}$**= An.**

**\*The set An is clearly the set of all those algebraic numbers which are yhe roots of polynomial equations of degree n with rational coefficients .**

**\*Since the union of a countable family of countable sets is countable, therefore, An is countable.**

**\*Let us now write** $\bigcup\_{n=1}^{\infty }A\_{n}$**= A.**

**\*A is clearly the set of algebraic numbers .**

**\*Since A is the union of a countable family of countable sets , and since the union of every countable family of countable sets is countable, therefore , A is countable.**

**\*Hence the set of algebraic numbers is countable.**

**DEFINITION 10.4**

**A real number is said to be transcendental if it is not algebraic.**

**THEOREM 10.9**

**The set of transcendental numbers is uncountable.**

**PROOF**

**\*Let T be the set of transcendental numbers and let A be the set of algebraic numbers.**

**\*If T be countable , then the set T**$∪$ **A will be countable .**

**\*Since , by , definition , T**$∪$ **A=R, and R is known to be uncountable, therefore , we have a contradiction .**

**\*Hence the set T must be uncountable.**

**NEIGHBOURHOODS**

**DEFINITION 2.1 (a)**

**A set N ⊂ R is said to be a neighbourhood of a point p**$\in $ **R if there exists an Є >0 such that ( p- Є, p+ Є) ⊂ N .**

**DEFINITION 2.1(b)**

**A set N ⊂ R is said to be a neighbourhood of a point p Є R if there exists an open interval (a,b)containing p and contained in N.**

**ILLUSTRATIONS**

**1.The open intervals (a,b) is a neighbourhood of each of its points.**

**2.R is a neighbourhood of each of its points.**

**3.G= (1,2)**$ ∪$ **(3,4) is a neighbourhood of each of its points.**

**4.The closed interval [a,b] is a neighbourhood of each point of (a,b) but is not a neighbourhood of the end points a and b.**

**5.The set Z of integers is not a neighbourhood of any of its points.It is obvious from the definition , that each point of R has at least one neighbourhood (R is a neighbourhood of each of its points )and that if N be a neighbourhood of a point p, then**

**p**$\in $ **N.**

**DEFINITION 2.2**

**If N is a neighbourhood of p , then we say that p is an interior point of N.**

**THEOREM 2.1**

**If M and N are neighbourhood of a point p, then M**$∩$ **N is also a neighbourhood of p.**

**PROOF**

**\*Since M and N are neighbourhoods of p , therefore , there exists Є1>0 and Є2>0 such that [p- Є1,p+ Є1) ⊂ M**

 **[p- Є2,p+ Є2) ⊂ N.**

 **and**

**If Є = min { Є1, Є2} , then**

 **(p- Є,p+ Є) ⊂( p- Є1,p+ Є1) ⊂M**

 **(p- Є,p+ Є) ⊂( p- Є2,p+ Є2) ⊂N**

**So that (p- Є,p+ Є) ⊂ M**$∩$ **N**

**and consequently M**$∩$ **N is a neighbourhood of p.**

**THEOREM 2.2**

**If M is a neighbourhood of a point p, and**

 **N⊃ M , then N is also a neighbourhood of p.**

**PROOF**

**\*Since M is a neighbourhood of p, therefore , for some Є>0 , we must have**

**(p- Є,p+ Є) ⊂M.**

**Since M ⊂N, therefore , it follows from above that ,**

**and consequently N is also a neighbourhood of p. (p- Є,p+ Є) ⊂N**

**OPEN SETS**

**DEFINITION 3.1(a).**

**A set of G ⊂ R is said to be open if it is a neighbourhood of each of its points.**

**DEFINITION 3.1(b)**

**A set G ⊂R is said to be open if for each p**$\in $ **G, there exists Є>0 such that (p- Є,p+ Є) ⊂G.**

**ILLUSTRATIONS**

**1.Every open interval (a,b) is an open set.**

**2.The interval [a,b) is not an open set; for it is not a neighbourhood of a.**

**3.The interval (a,b] is not an open set.**

**4.The closed interval [a,b] is not an open set.**

**5.R is an open set .For, if x be any point of R, then the open interval (x-1,x+1) ⊂ R and consequently R is a neighbourhood of x.Since x is any point of R , therefore ,R is a neighbourhood of each of its points.By definition 3.1(a) , it follows that R is an open set .**

**6.**$∅$ **is an open set .For , there is no point at all in** $∅$**, and consequently there is no point in**$∅$**, of which it is not a neighbourhood .This shows that** $∅$ **satisfies the condition of definition 3.1(a) trivially. Thus** $∅$ **is open.**

**7.The set (1,2)**$∪$ **(3,4) is open.**

**8.The open rays (a,**$\infty $**) and (-**$\infty $**, a) are open sets.**

**9.The closed rays [a,**$ \infty $**) and (-**$\infty $**, a] are not open sets.**

**THEOREM 3.1**

**The union of an arbitrary family of open sets is open.**

**PROOF**

**\*Let F be the union of an arbitrary family ℱ of open sets in R. To show that F is an open set , consider any p**$\in $ **F.**

**\*Since F is the union of members of ℱ , therefore , there must exist its an open set H⊂F such that p**$\in $ **H⊂F .**

**\*Since H is an open set and p**$ \in $ **H , therefore ,there must exist Є>0 such that**

**(p- Є,p+ Є) ⊂ H⊂F .**

**\*Again since [p- Є, p+ Є) is contained in F , therefore , F is a neighbourhood of p .**

**\*Since p is any point of F , therefore , it follows that F is a neighbourhood of each of its points, and consequently , F is an open set.**

**THEOREM 3.2**

**The intersection of two open sets is open**

**PROOF**

**\*Let G ⊂ R and H ⊂ R be two open sets .**

**\*If G**$∩$ **H =**$∅$**, then it is open.**

**\*If G**$∩$ **H**$\ne ∅$ **, let p be any point of G**$∩$ **H.**

**\*Now p**$ \in $ **G**$∩$ **H**$⇒$ **p** $\in $ **G and p**$ ∩$ **H,**

$⇒$ **G and H are neighbourhood of p,**

$⇒$ **G**$∩$ **H is a neighbourhood of p.**

**\*Since p is any point of G**$∩$ **H , therefore , it follows that G**$∩$ **H is a neighbourhood of each of its points, and consequently , G**$∩$ **H is an open set.**

**CLOSED SETS**

**DEFINITION 4.1.**

**A set F⊂R is said to be closed if its complement (that is ,R**$\~$**F ) is open.**

**ILLUSTRATIONS**

**1.Every closed interval [a,b] is a closed set.For, if F=[a,b], then**

**R**$\~$**F =(-**$ \infty $**,a)** $∪$ **(b,**$ \infty $**).Now each of the rays**

**(-**$ \infty $**,a) and (b,**$ \infty $**) is an open set ,and consequently their union is an open set.Since R**$\~$**F is an open set , therefore , F is a closed set.**

**2.The open interval (a,b) is not a closed set.For , if F = (a,b) , then**

**R**$\~$**F= (-**$ \infty $**,a)** $∪$ **(b,**$ \infty $**).Now R**$\~$**F is not an open set because a is a point of R**$\~$**F but R**$\~$**F is not a neighbourhood of a .Since R**$\~$**F is not open, therefore ,F is not a closed set.**

**3.Neither of the intervals [a,b) and (a,b] is a closed set .**

**4.The empty set** $∅$ **is a closed set . For**

**R**$\~∅$ **=R is an open set.**

**5.R is a closed set because its complement , namely** $∅$ **is an open set .**

**6.The set [1,2]**$ ∪$ **[3,4] is a closed set.**

**7.The closed ray (-**$\infty $**,a] is a closed set because its complement namely (a,**$ \infty $**) is an open set.**

**8.The closed ray [a,**$ \infty )$ **is a closed set because its complement namely (-**$\infty $**,a) is an open set.**

**9.The open rays (**$\infty $**,a) and (a,**$ \infty $**) are not closed sets.**

**THEOREM4.1**

**The intersection H of an arbitrary family** $δ$ **of closed sets is a closed set.**

**PROOF**

**\*Let** $δ$**\*={ R**$\~$**F : F**$\in δ$**}.**

**\*By De Morgan’s rule , R**$\~$**H is the union of the complement of members of** $δ$ **, that is , R**$\~$**H is the union of members of** $δ$**\*.**

**\*Since each member of** $δ$**\*is an open set , therefore , it follows that R**$\~$**H is an open set , and consequently , H is a closed set.**

**THEOREM:4.2**

**The union of two closed sets is a closed set.**

**PROOF**

**\*Let P⊂ R and Q⊂R be two closed sets, and let F=P**$∪$**Q .**

**\*Then R**$\~$**F=R**$\~$**(P**$∪$**Q)=( R**$\~$**P)**$∩$**( R**$\~$**Q).**

**\*Since P and Q are closed sets , therefore , R**$\~$**P and R**$\~$**Q are open sets .**

**\*Since the intersection of two open sets is an open set, therefore, ( R**$\~$**P)**$∩$**( R**$\~$**Q) is an open set , that is , R**$\~$**F is an open set , and consequently , F is a closed set.**

**LIMIT POINT OF A SET**

**DEFINITION 5.1(a).**

**A point p**$\in $ **R is said to be a limit point (or an accumulation point) of a set S⊂R if every neighbourhood of p contains a point of S different from p.**

 **In symbols , the above definition means that a point p**$\in R$ **is a limit point of S⊂R iff for each neighbourhood N of p,**

 **(N**$∩$**S)**$ \~$**{p}**$\ne ∅$**.**

**DEFINITION 5.1(b).**

**A point p**$\in $**R is said to be a limit point (or accumulation point) of a set S⊂R if for each Є>0 , the open interval (p- Є,p+ Є) contains a point of S other than p.**

**DEFINITION 5.2**

**The set of all limit points of a set S⊂R is called the derived set of S and is denoted by S'.**

**ILLUSTRATIONS**

**1.Every real number is a limit point of the set Q of all rational numbers.For ,if p be any real number whatever ,and Є>0 be given than (p- Є,p+ Є) contains infinitely many rational numbers and consequently , it contains at least one point of Q other than p.**

**Hence p is a limit point of Q .The derived set of Q is , therefore , R.**

**2.Every point of [0,1] is a limit point of the open interval (0,1).**

**3.The set of Z of integers has no limit point.**

**4.A finite set has no limit point.**

**EXISTENCE OF LIMIT POINTS OF A SET**

**THEOREM 5.1**

**BOLZANO – WEIERSTRASS THEOREM**

**Statement**

**Every infinite bounded set of real numbers has a limit point.**

**PROOF**

**\*Let S be an infinite bounded set, and let k and k' be its infimum and supremum respectively.**

**\*Let H be the set of real numbers having the following property : x in H iff it exceeds only a finite number of members of S.**

**\*The set H is clearly non empty for k**$ \in $ **H .**

**\*Also , H is bounded above , for no number greater than k' belongs to H .**

**\*By the completeness property , H must have a supremum , say p.**

**\*We shall show that p is a limit point of S.**

**\*Let Є>0 be given.**

**\*Since p is the supremum of H , therefore , there exist a member q of H such that q>p- Є**

**\*Since q**$ \in $**H , therefore , it exceeds only a finite number of members of S , and consequently , p- Є also exceeds only a finite number of members of S.**

**\*Also , since p is the supremum of H , therefore, p+ Є must exceed infinitely many members of S .**

**\*Now p- Є exceeds only finitely many members of S and p+ Є exceeds infinitely many members of S.**

**\*This means that (p- Є,p+ Є) contains infinitely many members of S , and consequently p must be a limit point of S.**

 **The above**